Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Minimum Energy Intercept of a Deorbiting Target

DENNIS E. FITZGERALD*
McDonnell Douglas Astronautics Co., Santa Monica, Calif.

THE problem considered in this paper is the derivation of optimal policy for the intercept of a deorbiting target. The optimum policy desired is one which minimizes the total energy required of the interceptor to achieve intercept. The control parameters available to effect the minimization are 1) the initial condition of the predicted interceptor position at the start of the engagement and 2) the acceleration of the interceptor during the engagement. The problem is first transformed into a discrete time problem in which the interceptor acceleration control becomes the sequence of discrete velocity increments. The total energy, assuming constant mass, is proportional to the sum of the velocity increments so that velocity costs may be considered rather than energy costs.

The control available to the target in deorbiting is assumed to consist of 1) the time to initiate constant acceleration retro forces and 2) the duration of retro, which is assumed to be constrained to be greater than t_{min} seconds and less than t_{max} seconds. These bounds were chosen to reflect a minimum change in velocity if the target is to re-enter the atmosphere and not "skip" back out, and an upper bound on velocity change due to fuel limitations. It will be assumed that the offense has full knowledge of the defense strategy and will use the target controls so as to maximize the velocity cost to the defense. The defense problem then becomes one of finding the control policy which minimizes the maximum total velocity that could possibly be required to achieve intercept. A dynamic programing approach is used for the solution.[1]

The target is assumed to be constrained to the orbit plane; however, the interceptor may be approaching from any direction. The assumption is made that the control accelerations (for interceptor and target) act normal to the relative velocity vector and that the intercept time remains fixed. Furthermore, it is assumed that the defense recognizes action taken by the offense instantaneously.

The control solution is obtained for two intervals. The first is the preretro interval. This solution defines the optimal control prior to the time the target initiates retro. The second is the postretro interval. This solution defines the optimal control for the interval after the target initiates retro. The control solution is presented in terms of the optimum predicted interceptor position (or aim point). The actual acceleration applied to the interceptor vehicle is that which maintains this optimal aim point.

The reduced problem in which the target burn time variation is zero ($t_{max} = t_{min}$) is considered first. This situation occurs in the larger problem when time to intercept is less than t_{min}. It is shown that this is also the form of the postretro initiation problem where the only option open to the target is the time to terminate retro and, once termination occurs, the intercept point is determined. Results for the reduced problem are used to solve the enlarged problem where the target has both control parameters at its disposal.

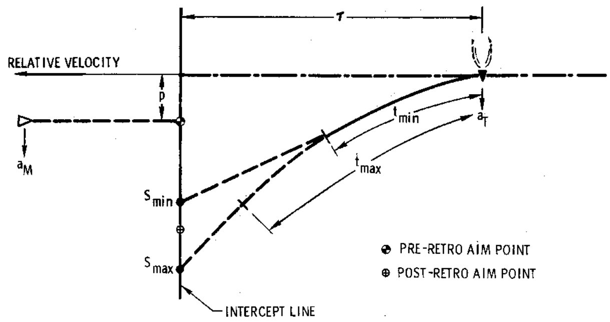

Fig. 1 Engagement geometry.

The engagement geometry is illustrated in Fig. 1. The intercept line is in the targets orbit plane and is normal to the interceptor-target relative velocity vector. The initial relative velocity vector is assumed to be of large magnitude compared to any changes in the relative velocity which may occur due to differential maneuvering between the target and interceptor. With these assumptions the intercept will occur at a fixed time from the start of the engagement.

The objective function to be minimized over the total time of the engagement T is the velocity expended by the interceptor to achieve intercept. If the total time of the engagement is divided into n intervals to form an n-stage engagement then the objective function is given by the sum of the velocity increments

$$J = \sum_{i=1}^{N} |\Delta v_i| \tag{1}$$

The motion of the predicted interceptor position p is described by the differential equation $\dot{p} = a_n \tau$ where a_M is the interceptor acceleration and τ is the time to intercept ($\tau = T - t$). Similarly the rate of change of the maximum target displacement, S_{max}, if the target does not retro is given by $\dot{S}_{max} = -a_T \tau$. Integrating these equations over one time interval Δt results in the following state transition equations

$$S_{max_{k+1}} = S_{max_k} - a_T \Delta t T_k \qquad p_{k+1} = p_k + \Delta v T_k \tag{2}$$

where T_k is defined as $\tau - \Delta t/2$.

If the retro duration of the evader is known ($t_{max} = t_{min}$) the postretro control solution is simply an impulse of velocity to cause $p = S_{max}$. The impulsive velocity, u, required is $u = [S_{max} - p]/\tau$. Assume that the engagement has preceded to stage k and the target has not initiated retro as yet. The cost to the defense if retro commences at stage k is

$$J_k = \sum_{i=1}^{k} |\Delta v_i| + u_k \tag{3}$$

It has been assumed that the offense will choose the time of retro initiation so as to cause the largest velocity expenditure on the part of the defense. Therefore, to solve the preretro problem an initial condition, p_o, and a command policy (sequence of Δv_i) must be found which minimizes the maximum value of J. Let the function $f_n(p)$ be the minimax cost over the last n stages of the engagement starting at a given value for state p. The sequence of interceptor controls: $\Delta v_1, \Delta v_2, \Delta v_3 \ldots$ must be chosen

Presented as Paper 70-1019 at the AIAA Guidance, Control and Flight Mechanics Conference, Santa Barbara, Calif., August 17–19, 1970; submitted October 29, 1970; revision received October 1, 1971.
Index category: Launch Vehicle and Missile Guidance Systems.

* Section Chief, Guidance Implementation and Vehicle Requirements, Strategic Defense Guidance Branch.

so as to minimize the total velocity required J over all possible paths (which can be enforced by the evader) through the network of states. The minimax cost and the optimal control is determined by the solution of the equation.

$$f_n(p) = \max \{u, \min_{\Delta v} [\Delta v + f_{n-1}(p + T\Delta v)]\} \tag{4}$$

The variable x_n is defined as the optimal aim point. The optimal aim point, x_n is the initial predicted position of the interceptor, p, which will minimize the cost over the n remaining stages. The minimum cost is $f_n(x_n)$. If the interceptor initial condition p is less than x_n the target will retro. If p is greater than x_n the target will not initiate retro. The optimal interceptor control is

$$\begin{aligned} \Delta v_n &= (x_{n-1} - p)/t_n, \quad p \geqq x_n, \quad \text{no retro} \\ \Delta v_n &= u_n = (s_n - p)/t_n, \quad p < x_n, \quad \text{retro} \end{aligned} \tag{5}$$

and the cost is determined by substituting the optimal control Eq. (5) into Eq. (4).

Of special interest is the minimum cost path represented by the aim points $x_1, x_2 \ldots x_n$ and the corresponding minimum cost $f_n(x_n)$. These are given by the following recursive equations

$$x_n = 1/2[s_n + x_{n-1} - T_n f_{n-1}(x_{n-1})] \tag{6}$$

$$f_n(x_n) = f_{n-1}(x_{n-1}) + |x_n - x_{n-1}|/T_n \tag{7}$$

The minimum cost occurs when the aim points $p_1, p_2 \ldots p_n$ are chosen so as to make the cost equal over all possible paths. Thus, when the optimal aim point is maintained the impulse required to divert if the target does retro is given by

$$u_n = |s_n - x_n|/T_n \tag{8}$$

$$u_n = |x_n - x_{n-1}|/T_n + f_{n-1}(x_{n-1}) = f_n(x_n) \tag{9}$$

The optimal aim point $x(\tau)$ and the corresponding minimal velocity, $f(\tau)$, can be determined by converting the difference Eqs. (6) and (7) into differential equations. The time-to-intercept, τ is defined as $\tau = n\Delta t$. In the limit as Δt approaches zero the equations become

$$x(\tau) = s(\tau) - \tau f(\tau) \tag{10}$$

$$df/d\tau = (1/\tau)(dx/d\tau) \tag{11}$$

Combining Eqs. (10) and (11) and multiplying the result by the factor $\tau^{1/2}$ yields

$$\tau^{1/2} f = \int (1/2)\tau^{-1/2}(ds/d\tau)\, d\tau \tag{12}$$

where

$$\frac{ds}{d\tau} = \begin{cases} a_T \tau & \tau < t_{max} \\ a_T t_{max} & \tau > t_{max} \end{cases} \tag{13}$$

The solution is determined by the boundary condition $f(o) = 0$ thus

$$\begin{aligned} f(\tau) &= a_T (1/3)\tau \qquad \tau < t_{max} \\ f(\tau) &= a_T t_{max} - (2/3) a_T t_{max} (t_{max}/\tau)^{1/2} \qquad \tau > t_{max} \end{aligned} \tag{14}$$

The optimal aim point is determined from Eq. (10). From these results it can be seen that the cost to the interceptor (total velocity expended) is equal to one-third of the velocity expended by the target if the duration of the engagement is less than the duration of the retro. As the duration of the engagement becomes very long with respect to the duration of retro the cost approaches the total velocity ($a_T t_{max}$) capability of the target.

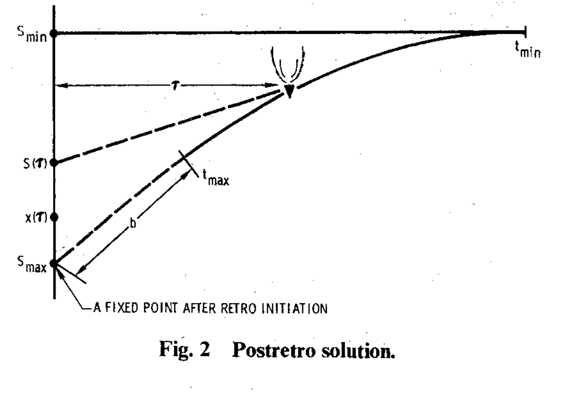

Fig. 2 Postretro solution.

In the unknown burn time case the target may retro at any time to intercept τ and may subsequently terminate retro at any time providing the time duration of the retro is greater than t_{min} but less than t_{max}. If retro occurs at $\tau < t_{min}$ the post-retro solution and preretro solution is the same as for the unknown burn time case. To determine the solution for the cost and optimal preretro aim point for the case of $\tau > t_{min}$ it is first necessary to determine the optimal postretro control and minimum cost. The postretro engagement geometry is illustrated in Fig. 2. Note that once the retro is initiated the points S_{min} and S_{max} become fixed. During the postretro portion of the engagement after the retro duration has exceeded t_{min} the target may terminate retro at any time to intercept τ thus terminating the state $s(\tau)$ which lies between S_{min} and S_{max}. The optimal control if the target terminates retro is an impulse of velocity, $u = |s - p|/\tau$ which will cause the predicted interceptor position p to be equal to $s(\tau)$.

The solution for the optimal aim point and the minimax cost is again determined by the solution to Eq. (12) when s is defined in this case as a positive distance from the fixed point S_{max}. As in the known burn time case $ds/d\tau = a_T \tau$. The solution depends on the boundary condition. If retro occurs at $t_{min} < \tau < t_{max}$. The retro duration may last until intercept. For this case the appropriate boundary condition is $f(o) = 0$. The postretro cost is $f(\tau) = (\frac{1}{3}) a_T \tau$ and the optimal interceptor control is given by the aim point $x(\tau) = (\frac{1}{3})(\frac{1}{2}) a_T \tau^2$ where $x(\tau)$ defined relative to the point S_{max}. If retro is initiated at $\tau > \tau_{max}$ the solution for the postretro cost and aim point is determined by the boundary condition $f(b) = 0$ (see Fig. 2) thus

$$f(\tau) = \tfrac{1}{3}(a_T)[\tau - b(b/\tau)^{1/2}] \tag{15}$$

$$x(\tau) = a_T \tfrac{1}{2}[\tau^2 - b^2] - \tfrac{1}{3} a_T [\tau^2 - b(b\tau)^{1/2}] \tag{16}$$

The solution for the preretro aim point and the total cost of the engagement is obtained by making a modification to the recursive equation for the optimal aim point (Eq. 6) to include the cost of the postretro control if the target initiates retro. The minimum cost function for the total engagement is indicated by the capital letter "F" instead of the small letter f which is used to indicate the cost for the postretro control interval only.

Similarly, capital letters X and S are used to indicate respectively the optimal preretro aim point and the divert point when retro is initiated. Using this nomenclature Eq. (6) becomes

$$X_n = (1/2)[S_n + X_n + T_n f(n\Delta - t_{min}) - T_n F_{n-1}(X_{n-1})] \tag{17}$$

and the recursive relationship for the minimum total cost is

$$F_n(X_n) = F_{n-1}(X_{n-1}) + |X_n - X_{n-1}|/T_n \tag{18}$$

The solution is again determined by allowing Δt to shrink to zero which yields the following continuous equations

$$X(\tau) = S(\tau) - \tau F(\tau) + \tau f(\tau - t_{min}) \tag{19}$$

$$dF/d\tau = (1/\tau)(dX/dt) \tag{20}$$

Combining these equations results in the following differential equation for the minimum cost

$$\tau^{1/2} F = \int \frac{1}{2}\tau^{-1/2}\frac{dS}{d\tau} + \frac{1}{2}\tau^{1/2}\frac{df}{d\tau}(\tau - t_{min}) + \frac{1}{2}\tau^{-1/2} f(\tau - t_m)\, d\tau \tag{21}$$

Table 1 Summary

Interval	Preretro aim point
$\tau \leqq t_{min}$	$(1/3)(1/2)a_T \tau^2$
$t_{max} = t_{min} < \tau$	$a_T t_{max}(\tau - (1/2)t_{max} - (1/3)(t_{max}\tau))^{1/2}$
$t_{min} < \tau < t_{max}$	$(2/9)a_T \tau^2 - (1/6)t_{min}^2 + (1/9)a_T t_{min}(t_{min}\tau)^{1/2}$
	Postretro aim point[a]
$\tau_r \leqq t_{min}$, $0 < \tau \leqq \tau_r$	$(1/2)a_T \tau_r^2$
$t_{max} = t_{min} < \tau_r$, $0 < \tau < \tau_r$	$a_T t_{max}(\tau_r - (1/2)t_{max})$
$t_{min} < \tau_r < t_{max}$, $(\tau_r - t_{min}) < \tau < \tau_r$	$(1/2)a_T \tau_r^2 - (1/3)(1/2)a_T(\tau_r - t_{min})^2$
$t_{min} < \tau_r < t_{max}$, $\tau < (\tau_r - t_{min})$	$(1/2)a_T \tau_r^2 - (1/3)(1/2)a_T \tau^2$
	Cost[b]
$T \leqq t_{min}$	$(1/3)a_T T$
$t_{max} = t_{min} < T$	$a_T t_{max}[1 - (2/3)(t_{max}/T)^{1/2}]$
$t_{min} < T < t_{max}$	$(4/9)a_T T - (1/9)a_T t_{min}(t_{min}/T)^{1/2}$

[a] τ_r time to intercept when retro occurred.
[b] T total engagement duration.

If retro occurs at $t_{min} < \tau < t_{max}$ the postretro cost is given by $f(\tau - t_{min}) = \frac{1}{3}a_T(\tau - t_{min})$ and the optimum divert point is $S(\tau) = \frac{1}{2}a_T\tau^2 - \frac{1}{3}\frac{1}{2}a_T(\tau - t_{min})^2$. The solution for the total cost is then the solution of

$$\tau^{1/2}F = \int (2/3)\tau^{1/2}a_T\, d\tau \tag{22}$$

The boundary condition which determines the solution is the total cost for the case of $\tau = t_{min}$ in which case there is no postretro cost and the total engagement cost for $t_{min} < \tau < t_{max}$ is

$$F(\tau) = (4/9)a_T\tau - (1/9)a_T t_{min}(t_{min}/\tau)^{1/2} \tag{23}$$

The optimal solutions for the preretro aim point, the postretro aim point and the total cost (total interceptor velocity required) of the engagement are summarized in Table 1.

Reference

[1] Bellman, R. E., "Dynamic Programming," Princeton University Press, Princeton, N.J., 1957, pp. 283–309.

Simple Waves in One-Dimensional Unsteady Nonequilibrium Dissociative Gasdynamics

RAMA SHANKAR*
Indian Institute of Technology, New Delhi, India

I. Introduction

STUDY of simple waves in general relaxation hydrodynamics has been the subject of investigation by Coburn.[1–3] Using the set of equations given by Stupochenko and Stakhanov,[4] Coburn has arrived at a number of interesting theorems. As the study has been quite general some of the theorems escaped physical interpretations. Accordingly, we have in this Note, made an attempt to give physical interpretation to one of the important theorems due to Coburn by specializing it to the case of ideal dissociative gasdynamics.

II. Simple Waves in One-Dimensional Unsteady Motion

Following Coburn,[2] the characteristic manifolds for the system of equations given in Ref. 2, can be expressed as

$$X = I\hat{C}^2 - J \tag{1}$$

where

$$I = (Fk + GT)K + 2\bar{q}_1 F \tag{2a}$$

$$J = (FkC_\infty^2 + GTC_0^2)K + 2\bar{q}_1 FC_\infty^2 \tag{2b}$$

$$\hat{C}^2 = 1/\phi(\phi_0 + v^j\phi_j)^2, \quad L = \phi_0 + v^j\phi_j \tag{2c†}$$

The bicharacteristics of the same system are determined by

$$dx^j/d\sigma = \tfrac{1}{2}\partial X/\partial\phi_j = ILv^j - J\phi_j \tag{3}$$

$$dx^0/d\sigma = \tfrac{1}{2}\partial X/\partial\phi_0 = IL = \pm(IJ\phi)^{1/2} \tag{4}$$

Equations (3) and (4) yield

$$dx^j/dx^0 = v^j + \hat{C}n^j \tag{5}$$

where $\hat{C}$ is given by Eq. (1) and n^j, the space-unitized normal to the characteristic space-time manifolds S_3, is given by

$$n^j = \phi^j/\phi^{1/2} \tag{6}$$

Received February 22, 1971.
* A-Lecturer, Department of Mathematics.

† Symbols used in Eq. (2) have the same meaning as in Ref. 2.

If α, β be the parameters along the two families of bicharacteristics then, for the one-dimensional unsteady case, Eq. (5) can further be written as

$$\partial x/\partial\alpha = (u + \hat{C})(\partial t/\partial\alpha), \quad \partial x/\partial\beta = (u - \hat{C})(\partial t/\partial\beta) \tag{7}$$

For the theory of simple waves in the x, t plane we use the following definition: a family of simple waves consists of a family of bicharacteristics (with parameter, β = variable or α = constant, along each curve of the family) such that

$$\partial q/\partial\beta = \partial S/\partial\beta = \partial\rho/\partial\beta = 0 \tag{8a}$$

$$\partial v_j/\partial\beta = 0 \tag{8b}$$

Next, we consider those consequences of energy equations, the definition of rate equation[2] and the geometry of bicharacteristics Eqs. (1) and (7) and a relation of the type

$$\partial\alpha/\partial t = (\hat{C} - u)/2\hat{C}(\partial t/\partial\alpha)^{-1}, \quad \partial\alpha/\partial x = 1/2\hat{C}(\partial t/\partial\alpha)^{-1} \tag{9}$$

which are valid in both the aforementioned cases. The energy and rate equations in terms of the derivatives of S and q with respect to α can be expressed as

$$KT(dS/d\alpha) = 2\bar{q}^2(\partial t/\partial\alpha) \tag{10}$$

$$dq/d\alpha = 2\bar{q}(\partial t/\partial\alpha) \tag{11}$$

and elimination of $\partial t/\partial\alpha$ from Eqs. (10) and (11) yields

$$KT(dS/d\alpha) = \bar{q}(dq/d\alpha) \tag{12}$$

Further substituting the value of $\bar{q}/K = -(\partial e/\partial q)$, the Eq. (12) can be written as

$$de/dq = -T(dS/dq) \tag{13}$$

This fact can be stated in the form of the following theorem.

Theorem 1

If the rate of change of internal energy with respect to the relaxation variable is positive (negative) then as one moves from one simple wave to another, entropy decreases (increases) as relaxation variable increases.

Now, using the chain rule for differentiation, the definition Eq. (8), the expressions for $\partial\alpha/\partial t$, $\partial\alpha/\partial x$ from Eq. (9), the values of A, B, C as given in Ref. 1, and Eq. (12), the equation of continuity and motion for simple waves under consideration can be written as

$$\hat{C}(d\rho/d\alpha) + \rho(du/d\alpha) = 0 \tag{14}$$

$$\rho\hat{C}(du/d\alpha) + A\rho(d\rho/d\alpha) + ((B\bar{q}/KT) + C)dq/d\alpha = 0 \tag{15}$$

As in the equilibrium gasdynamics, Eq. (14) can be interpreted as follows.

Theorem 2

In both cases, as one moves from one simple wave to another, the density increases (decreases) as particle speed decreases (increases).

Further eliminating $d\rho/d\alpha$ from Eqs. (14) and (15) and using Eq. (12) we get

$$du/dq = -\hat{C}/(\rho(\hat{C}^2 - A))(B(dS/dq) + C) \tag{16}$$

The result Eq. (16) has been interpreted by Coburn[2] in the form of the following.

Theorem 3

If the rate of increase of internal energy or entropy with respect to the relaxation variable is such that $[C + B(dS/dq)]$ is positive then: 1) in the supersonic case $\hat{C}^2 > A$, the particle speed, u, is decreasing as the relaxation variable increases; and 2) in the subsonic case $\hat{C}^2 < A$, the particle speed, u is increasing as the relaxation variable increases.